THREE-DIMENSIONAL HOMOGENEOUS GENERALIZED RICCI SOLITONS

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ABSTRACT. We study three-dimensional generalized Ricci solitons, both in Riemannian and Lorentzian settings. We shall determine their homogeneous models, classifying left-invariant generalized Ricci solitons on three-dimensional Lie groups.

1. Introduction

Generalized Ricci solitons were recently introduced in [15]. A *generalized Ricci soliton* is a pseudo-Riemannian manifold (M, g) admitting a smooth vector field X, such that

$$\mathcal{L}_X g + 2\alpha X^\flat \odot X^\flat - 2\beta Ric = 2\lambda g, \tag{1.1}$$

for some real constants α, β, λ, where $\mathcal{L}_X$ denotes the Lie derivative in the direction of X, $X^\flat$ denotes a 1-form such that $X^\flat(Y) = g(X, Y)$ and Ric is the Ricci tensor.

For particular values of the constants α, β, λ, several important equations occur as special cases of equation (1.1). In particular, one has:

(K) the *Killing vector field equation* when $\alpha = \beta = \lambda = 0$;

(H) the *homothetic vector field equation* when $\alpha = \beta = 0$;

(RS) the *Ricci soliton equation* when $\alpha = 0$ and $\beta = 1$ [7].

(E-W) a special case of the *Einstein-Weyl equation* in conformal geometry when $\alpha = 1$ and $\beta = -\frac{1}{n-2}$ $(n > 2)$ [2];

(PS) the equation for a *metric projective structure* with a *skew-symmetric Ricci tensor* representative in the projective class when $\alpha = 1$, $\beta = -\frac{1}{n-1}$ and $\lambda = 0$ [17];

(VN-H) the *vacuum near-horizon geometry equation* of a spacetime when $\alpha = 1$ and $\beta = \frac{1}{2}$, with λ playing the role of the cosmological contant [10].

Equation (1.1) corresponds to an overdetermined system of partial differential equations of finite type. The study of this system was undertaken in the fundamental paper [15]. Explicit solutions were determined in [15] in the two-dimensional case. For the three-dimensional case, the authors restricted in [15] to the case with $\alpha = 0$. Note that as already pointed out in [15], for $\alpha = 0 \neq \beta$, rescaling the vector field X to $-\frac{1}{\beta}X$, equation (1.1)

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reduces to the Ricci soliton equation. We also observe that a trivial solution of (1.1) is given by $X = 0$ and $\beta = \lambda = 0$, so we shall always exclude this solution.

The aim of this paper is to determine the three-dimensional homogeneous models of generalized Ricci solitons. A connected, complete and simply connected three-dimensional homogeneous manifold, if not symmetric, is isometric to some Lie group equipped with a left-invariant metric (see [18] for the Riemannian case and [3] for the Lorentzian one). Moreover, with the obvious exceptions of $\mathbb{R} \times \mathbb{S}^2$ (Riemannian) and $\mathbb{R}_1 \times \mathbb{S}^2$ (Lorentzian), three-dimensional connected simply connected symmetric spaces are also realized in terms of suitable left-invariant metrics on Lie groups [4]. For this reason, we shall consider three-dimensional Lie groups, equipped with a left-invariant metric (either Riemannian or Lorentzian). We shall specify our study to solutions of (1.1) determined by a left-invariant vector field X. In this way, (1.1) will be transformed into a system of algebraic equations, which we can solve, obtaining a complete classification of three-dimensional left-invariant generalized Ricci solitons, and determining several new solutions of (1.1). We recall that the study of three-dimensional Ricci solitons already showed some interesting differences arising between the Riemannian case (for which left-invariant solutions do not occur [8]) and the Lorentzian one, where several left-invariant solutions exist [1]. Also for the broader class of generalized Ricci solitons, interesting differences show up between the Riemannian and the Lorentzian cases. Calculations have been checked by means of *Maple 16* ©.

2. 3D Riemannian left-invariant generalized Ricci solitons

Three-dimensional Riemannian Lie groups were classified in [13]. We shall treat separately the unimodular and non-unimodular cases.

2.1. **Unimodular case.** Let G be a connected three-dimensional Lie group with a left-invariant Riemannian metric. Choose an orientation for the Lie algebra $\mathfrak{g}$ of G, so that the cross product $\times$ is defined on $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ is unimodular if and only if the endomorphism L, defined by $[Z, Y] = L(Z \times Y)$, is self-adjoint ([13],[12]). Therefore, $\mathfrak{g}$ admits an orthonormal basis $\{e_1, e_2, e_3\}$ of eigenvectors for L, so that

$$[e_1, e_2] = Ce_3, \qquad [e_2, e_3] = Ae_1, \qquad [e_3, e_1] = Be_2, \tag{2.1}$$

for some real constants A, B, C. Explicitly, depending on the sign of A, B, C, the Lie group G is isomorphic to one of the cases listed in the following Table I.

Lie group	(A, B, C)
$SU(2)$	$(+,+,+)$
$\widetilde{SL}(2,\mathbb{R})$	$(+,+,-)$
$\widetilde{E}(2)$	$(+,+,0)$
$E(1,1)$	$(+,-,0)$
H_3	$(+,0,0)$
$\mathbb{R}^3$	$(0,0,0)$

Table I: 3D unimodular Riemannian Lie groups

In the above Table I and throughout the paper, $\widetilde{SL}(2,\mathbb{R})$ will denote the universal covering of $SL(2,\mathbb{R})$, $\widetilde{E}(2)$ the universal covering of the group of rigid motions in the Euclidean two-space, $E(1,1)$ the group of rigid motions of the Minkowski two-space and H_3 the Heisenberg group.

The description of the Ricci curvature with respect to the basis $\{e_1,e_2,e_3\}$ is well known (see again [13]). We have:

$$Ric = \begin{pmatrix} \frac{1}{2}(A^2-B^2-C^2)+BC & 0 & 0 \\ 0 & \frac{1}{2}(B^2-A^2-C^2)+AC & 0 \\ 0 & 0 & \frac{1}{2}(C^2-A^2-B^2)+AB \end{pmatrix}.$$

In particular, it is easily seen that the left-invariant metric is Einstein (equivalently, of constant sectional curvature, since we are in dimension three) if and only if either $A=B=C$, $A-B=C=0$, $A=B-C=0$ or $A-C=B=0$.

We now consider an arbitrary vector field X, that is, $X=X_ie_i\in\mathfrak{g}$, for some real constants X_1,X_2,X_3. Then, by (2.1), we get

$$\mathcal{L}_X g = \begin{pmatrix} 0 & (A-B)X_3 & (C-A)X_2 \\ (A-B)X_3 & 0 & (B-C)X_1 \\ (C-A)X_2 & (B-C)X_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1,e_2,e_3\}$. Moreover, since $\{e_1,e_2,e_3\}$ is orthonormal, for any vector $X=X_ie_i\in\mathfrak{g}$ we have $X^\flat\odot X^\flat(e_i,e_j)=X_iX_j$. Therefore, equation (1.1) becomes the following system of algebraic equations:

$$\begin{cases} 2\alpha X_1^2-\beta(A^2-B^2-C^2+2BC)=2\lambda, \\ 2\alpha X_2^2+\beta(A^2-B^2+C^2-2AC)=2\lambda, \\ 2\alpha X_3^2+\beta(A^2+B^2-C^2-2AB)=2\lambda, \\ (A-B)X_3+2\alpha X_1X_2=0, \\ (C-A)X_2+2\alpha X_1X_3=0, \\ (B-C)X_1+2\alpha X_2X_3=0. \end{cases} \tag{2.2}$$

By standard calculations we obtain the solutions of (2.2), proving the following result.

Theorem 2.1. *Let $\mathfrak{g}$ denote a three-dimensional unimodular Riemannian Lie algebra, as described by* (2.1) *with respect to a suitable orthonormal basis $\{e_1,e_2,e_3\}$. Then, up to a renumeration of e_1,e_2,e_3, the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}$ are the following:*

(1) $A=B=C$, $\alpha=0$, $\lambda=-\frac{1}{2}\beta A^2$, for all β and X: when $A=B=C$ (case of constant sectional curvature on $SU(2)$), all vectors in $\mathfrak{g}$ are Killing.

(2) $A=B=C$, $\lambda=-\frac{1}{2}\beta A^2$, $X=0$, for all α,β: the metric is Einstein.

(3) $A=B-C=0$, $\lambda=0$, $X=0$, for all α,β: the metric is flat.

(4) $A \neq B = C$, $\lambda = \frac{1}{2}\beta A(A-2C)$, $X_1 = \pm\sqrt{\frac{\beta A(A-C)}{\alpha}}$, $X_2 = X_3 = 0$, *for any* α, β *such that* $\alpha\beta A(A-C) > 0$.

Solutions *(1)-(3)* are somewhat "trivial", as they correspond to cases of metrics of constant sectional curvature and could be expected. On the other hand, by *(4)* we have solutions when just two between A, B, C coincide. By Table I, this yields nontrivial left-invariant generalized Ricci solitons on $SU(2)$, $\widetilde{SL}(2,\mathbb{R})$, $\widetilde{E}(2)$, H_3. Observe that $\alpha \neq 0$ in case *(4)*, while cases *(1)-(3)* are Einstein. Thus, no (nontrivial) left-invariant Ricci solitons occur in three-dimensional Riemannian Lie groups, coherently with the results of [8].

Finally, we remark that in solution *(4)*, if $\lambda = 0$ then $A = 2C$ and so, necessarily $\alpha\beta > 0$, which excludes the possibility of solutions of (PS). On the other hand, if (E-W) holds for a three-dimensional manifold, then $\alpha\beta = -1$. Therefore, condition $\alpha\beta A(A-C) > 0$ in *(4)* yields $A(A-C) < 0$. Similarly, in the case of the vacuum near-horizon geometry equation (VN-H) (considered in [15] both for Riemannian and Lorentzian two-manifolds), we have $\alpha\beta = 1$, so that condition $\alpha\beta A(A-C) > 0$ in *(4)* yields $A(A-C) > 0$. Taking into account the above Table I, we then have the following.

Corollary 2.2. *Three-dimensional Riemannian Lie group $SU(2)$ gives solutions to the special Einstein-Weyl equation* (E-W). *Three-dimensional Riemannian Lie groups $SU(2)$, $\widetilde{SL}(2,\mathbb{R})$, $\widetilde{E}(2)$, H_3 give solutions to the vacuum near-horizon geometry equation* (VN-H).

2.2. **Non-unimodular case.** Let now $\mathfrak{g}$ denote a three-dimensional non-unimodular Riemannian Lie algebra. Then, its unimodular kernel $\mathfrak{u}$ is two-dimensional. Choosing an orthonormal basis $\{e_1, e_2, e_3\}$ so that e_1 is orthogonal to $\mathfrak{u}$ and $[e_1, e_2], [e_1, e_3]$ are mutually orthogonal [13], the bracket product is described by

(2.3)
$$[e_1, e_2] = Ae_2 + Be_3, \quad [e_1, e_3] = Ce_2 + De_3, \quad [e_2, e_3] = 0, \quad A + D \neq 0,\ AC + BD = 0,$$

for some real constants A, B, C, D.
With respect to the basis $\{e_1, e_2, e_3\}$, the Ricci curvature is described by (see [13])

$$Ric = \begin{pmatrix} -A^2 - \frac{1}{2}B^2 - \frac{1}{2}C^2 - D^2 - BC & 0 & 0 \\ 0 & -A^2 - \frac{1}{2}B^2 + \frac{1}{2}C^2 - AD & 0 \\ 0 & 0 & -D^2 + \frac{1}{2}B^2 - \frac{1}{2}C^2 - AD \end{pmatrix}.$$

In particular, the left-invariant metric is of constant sectional curvature if and only if $A - D = B + C = 0$.

For an arbitrary left-invariant vector field $X = X_i e_i \in \mathfrak{g}$, we have

$$\mathcal{L}_X g = \begin{pmatrix} 0 & AX_2 + CX_3 & BX_2 + DX_3 \\ AX_2 + CX_3 & -2AX_1 & -(B+C)X_1 \\ BX_2 + DX_3 & -(B+C)X_1 & -2DX_1 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$, and we have again $X^\flat \odot X^\flat(e_i, e_j) = X_i X_j$. Hence, equation (1.1) now gives

$$\begin{cases} 2\alpha X_1^2 + \beta(2A^2 + B^2 + C^2 + 2D^2 + 2BC) = 2\lambda, \\ -2AX_1 + 2\alpha X_2^2 + \beta(2A^2 + B^2 - C^2 + 2AD) = 2\lambda, \\ -2DX_1 + 2\alpha X_3^2 + \beta(2D^2 - B^2 + C^2 + 2AD) = 2\lambda, \\ AX_2 + CX_3 + 2\alpha X_1 X_2 = 0, \\ BX_2 + DX_3 + 2\alpha X_1 X_3 = 0, \\ -(B + C)X_1 + 2\alpha X_2 X_3 = 0. \end{cases} \tag{2.4}$$

We now solve (2.4) and list its different solutions, proving the following result.

Theorem 2.3. *Let $\mathfrak{g}$ denote a three-dimensional non-unimodular Riemannian Lie algebra, as described by* (2.3) *with respect to a suitable orthonormal basis $\{e_1, e_2, e_3\}$. Then, the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}$ are the following:*

(1) $A - D = B + C = 0$, $\lambda = (2\alpha^2\beta + \alpha)X_1^2$, $X_1 = -\frac{A}{\alpha}$, $X_2 = X_3 = 0$, for all $\alpha \neq 0$ and β (constant sectional curvature).

(2) $C = D = 0$, $\lambda = \frac{1}{2}\beta(2A^2 + B^2)$, $X_1 = X_2 = 0$ and $X_3 = \pm\sqrt{\frac{\beta(A^2+B^2)}{\alpha}}$, for all α and β satisfying $\alpha\beta > 0$.

(3) $A - D = B + C = 0$, $\lambda = 2\beta A^2$, $X = 0$, for all α, β: the metric is Einstein.

(4) $B = C = 0$, $\alpha = -\frac{A^2+D^2}{\beta(A+D)^2} \neq 0$, $\lambda = 0$, $X_2 = X_3 = 0$, for any $\beta \neq 0$ and $X_1 = \beta(A + D)$.

(5) $A = D$, $B = C = 0$, $\lambda = A^2(\frac{1}{\alpha} + 2\beta)$, for any $\alpha \neq 0$ and β, with $X_2 = X_3 = 0$ and $X_1 = -\frac{A}{\alpha}$ (constant sectional curvature).

It is easy to check that the above cases *(1)* and *(5)* are compatible with (PS), since for $\alpha = 1$ and $\beta = -\frac{1}{2}$ we have $\lambda = 0$ in both *(1)* and *(5)*). Moreover, solutions of the form *(4)* compatible with (PS) are a special case of (1), while all cases *(1), (4), (5)* yield solutions compatible with (E-W). Finally, cases *(1)*, *(2)* and *(5)* are compatible with (VN-H). Thus, we have the following.

Corollary 2.4. *Three-dimensional non-unimodular Riemannian Lie groups give solutions to the special Einstein-Weyl equation* (E-W)*, to the vacuum near-horizon geometry equation* (VN-H) *and (in the case of constant sectional curvature) to the equation* (PS) *for a metric projective structure with a skew-symmetric Ricci tensor representative.*

3. 3D Lorentzian left-invariant unimodular generalized Ricci solitons

Let now $\times$ denote the Lorentzian vector product on the Minkowski space $\mathbb{R}^3_1$, induced by the product of the para-quaternions ($e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2$, for a pseudo-orthonormal basis e_1, e_2, e_3, with e_3 time-like. The Lie bracket $[,]$ defines the corresponding Lie algebra $\mathfrak{g}$, which is unimodular if and only if the endomorphism L,

defined by $[Z,Y]=L(Z\times Y)$, is self-adjoint [16]. Differently from the Riemannian case, in Lorentzian settings L can assume four different standard forms (*Segre types*), giving rise to four classes of three-dimensional unimodular Lorentzian Lie algebras:

$\mathfrak{g}_1$: L is of Segre type $\{3\}$, that is, its minimal polynomial has a triple root.

$\mathfrak{g}_2$: L is of Segre type $\{1z\bar{z}\}$, that is, it has two complex conjugate eigenvalues.

$\mathfrak{g}_3$: L is of Segre type $\{11,1\}$, that is, diagonalizable.

$\mathfrak{g}_4$: L is of Segre type $\{21\}$, that is, its minimal polynomial has a double root.

We shall treat these cases separately.

3.1. **Lie algebra $\mathfrak{g}_1$.** There exists a pseudo-orthonormal basis $\{e_1,e_2,e_3\}$, with e_3 time-like, such that

$$\mathfrak{g}_1: \quad \begin{aligned} &[e_1,e_2]=Ae_1-Be_3,\\ &[e_1,e_3]=-Ae_1-Be_2,\\ &[e_2,e_3]=Be_1+Ae_2+Ae_3, \qquad A\neq 0. \end{aligned} \tag{3.1}$$

If $B\neq 0$, then $G=\widetilde{SL}(2,\mathbb{R})$, while $G=E(1,1)$ when $B=0$.

The curvature of Lorentzian Lie algebra $\mathfrak{g}_1$ was completely determined in [4]. In particular, with respect to $\{e_i\}$, the Ricci tensor is described by

$$Ric=\begin{pmatrix} -\frac{1}{2}B^2 & -AB & AB\\ -AB & -2A^2-\frac{1}{2}B^2 & 2A^2\\ AB & 2A^2 & -2A^2+\frac{1}{2}B^2 \end{pmatrix},$$

and the left-invariant metric is never Einstein.

For a left-invariant vector field $X=X_ie_i\in\mathfrak{g}$, we have

$$\mathcal{L}_X g=\begin{pmatrix} 2A(X_2-X_3) & -AX_1 & AX_1\\ -AX_1 & 2AX_3 & -A(X_2+X_3)\\ AX_1 & -A(X_2+X_3) & 2AX_2 \end{pmatrix}$$

with respect to the basis $\{e_1,e_2,e_3\}$, and $X^\flat\odot X^\flat(e_i,e_j)=\varepsilon_i\varepsilon_jX_iX_j$ for all indices i,j, where $\varepsilon_1=\varepsilon_2=-\varepsilon_3=1$ corresponds to the causal character of e_1,e_2,e_3. Therefore, equation (1.1) now becomes

$$\begin{cases} 2A(X_2-X_3)+2\alpha X_1^2+\beta B^2=2\lambda,\\ 2AX_3+2\alpha X_2^2+\beta(4A^2+B^2)=2\lambda,\\ 2AX_2+2\alpha X_3^2+\beta(4A^2-B^2)=-2\lambda,\\ -AX_1+2\alpha X_1X_2+2\beta AB=0,\\ AX_1-2\alpha X_1X_3-2\beta AB=0,\\ -A(X_2+X_3)-2\alpha X_2X_3-4\beta A^2=0. \end{cases} \tag{3.2}$$

Solving (3.2), we obtain the following.

Theorem 3.1. *Consider the three-dimensional unimodular Lorentzian Lie algebra $\mathfrak{g}_1$, as described by* (3.1) *with respect to a suitable pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time-like. Then, the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}_1$ are the following:*

(1) $\beta = 0$, $\lambda = 0$, $X_1 = 0$, $X_2 = X_3 = -\frac{A}{\alpha}$, *for all* $A (\neq 0)$, B *and* $\alpha \neq 0$.

(2) $\alpha = 0$, $\lambda = \frac{1}{2}\beta B^2$, $X_1 = 2\beta B$, $X_2 = X_3 = -2\beta A$, *for all* $A (\neq 0)$ *and* $\beta \neq 0$.

(3) $B = 0$, $\lambda = 0$, $X_1 = 0$, $X_2 = X_3 = \frac{(-1 \pm \sqrt{1 - 8\alpha\beta})A}{2\alpha}$, *for all* α, β *satisfying* $\alpha\beta \leq \frac{1}{8}$.

Solution *(2)* in Theorem 3.1 corresponds to the existence of Ricci solitons on this class of Lorentzian Lie algebras [1]. Solution *(3)*, requiring that $\alpha\beta \leq \frac{1}{8}$ and $\lambda = 0$, is incompatible with (VN-H), but compatible with (E-W) and (PS). These observations yield the following result.

Corollary 3.2. *Three-dimensional Lorentzian Lie group $E(1,1)$, with Lie algebra described by* (3.1)*, gives solutions to the special Einstein-Weyl equation* (E-W) *and the equation* (PS) *for a metric projective structure with a skew-symmetric Ricci tensor representative.*

3.2. **Lie algebra $\mathfrak{g}_2$.** There exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time-like, such that

$$\mathfrak{g}_2: \quad \begin{aligned} [e_1, e_2] &= -Ce_2 - Be_3, \\ [e_1, e_3] &= -Be_2 + Ce_3, \\ [e_2, e_3] &= Ae_1. \end{aligned} \qquad C \neq 0, \tag{3.3}$$

In this case, $G = \widetilde{SL}(2, \mathbb{R})$ if $A \neq 0$, while $G = E(1,1)$ if $A = 0$. With respect to $\{e_i\}$, the Ricci tensor is given by (see [4])

$$Ric = \begin{pmatrix} -\frac{1}{2}A^2 - 2C^2 & 0 & 0 \\ 0 & \frac{1}{2}A^2 - AB & C(A - 2B) \\ 0 & C(A - 2B) & -\frac{1}{2}A^2 + AB \end{pmatrix},$$

and the left-invariant metric is never Einstein.

The Lie derivative $\mathcal{L}_X g$ with respect to a vector $X = X_i e_i \in \mathfrak{g}$ is described by

$$\mathcal{L}_X g = \begin{pmatrix} 0 & -CX_2 + (A - B)X_3 & (B - A)X_2 - CX_3 \\ -CX_2 + (A - B)X_3 & 2CX_1 & 0 \\ (B - A)X_2 - CX_3 & 0 & 2CX_1 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$, and again $X^\flat \odot X^\flat(e_i, e_j) = \varepsilon_i \varepsilon_j X_i X_j$ for all indices i, j. Thus, equation (1.1) becomes

$$\begin{cases} 2\alpha X_1^2 + \beta(A^2 + 4C^2) = 2\lambda, \\ 2CX_1 + 2\alpha X_2^2 - \beta(A^2 - 2AB) = 2\lambda, \\ 2CX_1 + 2\alpha X_3^2 + \beta(A^2 - 2AB) = -2\lambda, \\ -CX_2 + (A - B)X_3 + 2\alpha X_1 X_2 = 0, \\ (B - A)X_2 - CX_3 - 2\alpha X_1 X_3 = 0, \\ -2\alpha X_2 X_3 - 2\beta C(A - 2B) = 0. \end{cases} \tag{3.4}$$

We then solve (3.4), obtaining the following classification result.

Theorem 3.3. *Consider the three-dimensional unimodular Lorentzian Lie algebra $\mathfrak{g}_2$, as described by* (3.3) *with respect to a suitable pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time-like. Then, the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}_2$ are given by*

(1) $A = -2B = \frac{4\alpha X_2 X_3}{3\varepsilon\sqrt{X_3^2 - X_2^2}}$, $C = \varepsilon\alpha\sqrt{X_3^2 - X_2^2}$, $\beta = -\frac{3}{8\alpha}$, $\lambda = \frac{\alpha(3X_2^4 - 10X_2^2X_3^2 + 3X_3^4)}{X_2^2 - X_3^2}$, $X_1 = -\frac{\varepsilon(X_2^2 + X_3^2)}{2\sqrt{X_3^2 - X_2^2}}$, *with* $\varepsilon = \pm 1$, *for all* $\alpha \neq 0$.

We observe that because of condition $\alpha\beta = -\frac{3}{8}$, the above solution is not compatible with any of equations (RS), (E-W), (PS) and (VN-H).

3.3. **Lie algebra $\mathfrak{g}_3$.** For a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time-like, of eigenvector of L, we have

$$\mathfrak{g}_3 : \quad [e_1, e_2] = -Ce_3, \quad [e_1, e_3] = -Be_2, \quad [e_2, e_3] = Ae_1. \tag{3.5}$$

The following Table II lists all the Lie groups G which admit a Lie algebra $\mathfrak{g}_3$, according to the different possibilities for A, B and C:

Lie group	(A, B, C)
$\widetilde{SL}(2, \mathbb{R})$	$(+, +, +)$
$\widetilde{SL}(2, \mathbb{R})$	$(+, -, -)$
$SU(2)$	$(+, +, -)$
$\widetilde{E}(2)$	$(+, +, 0)$
$\widetilde{E}(2)$	$(+, 0, -)$
$E(1, 1)$	$(+, -, 0)$
$E(1, 1)$	$(+, 0, +)$
H_3	$(+, 0, 0)$
H_3	$(0, 0, -)$
$\mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$	$(0, 0, 0)$

Table II: 3D Lorentzian Lie groups with Lie algebra $\mathfrak{g}_3$

Following [4] (or by direct calculation), the Ricci curvature of Lorentzian Lie algebra $\mathfrak{g}_3$, with respect to $\{e_i\}$, is described by

$$Ric = \begin{pmatrix} \frac{1}{2}(B^2 - A^2 + C^2) - BC & 0 & 0 \\ 0 & \frac{1}{2}(A^2 - B^2 + C^2) - AC & 0 \\ 0 & 0 & \frac{1}{2}(C^2 - A^2 - B^2) + AB \end{pmatrix},$$

and the left-invariant metric is Einstein (equivalently, of constant sectional curvature) if and only if either $A = B = C$, $A - B = C = 0$, $A - C = B = 0$ or $A = B - C = 0$. In the last three cases, the metric is flat.

For a left-invariant vector field $X = X_i e_i \in \mathfrak{g}$, we have

$$\mathcal{L}_X g = \begin{pmatrix} 0 & (A-B)X_3 & (C-A)X_2 \\ (A-B)X_3 & 0 & (B-C)X_1 \\ (C-A)X_2 & (B-C)X_1 & 0 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$. Finally, $X^\flat \odot X^\flat(e_i, e_j) = \varepsilon_i \varepsilon_j X_i X_j$ for all indices i, j. Hence, equation (1.1) now yields

$$\begin{cases} 2\alpha X_1^2 + \beta(A^2 - B^2 - C^2 + 2BC) = 2\lambda, \\ 2\alpha X_2^2 - \beta(A^2 - B^2 + C^2 - 2AC) = 2\lambda, \\ 2\alpha X_3^2 + \beta(A^2 + B^2 - C^2 - 2AB) = -2\lambda, \\ (A-B)X_3 + 2\alpha X_1 X_2 = 0, \\ (C-A)X_2 - 2\alpha X_1 X_3 = 0, \\ (B-C)X_1 - 2\alpha X_2 X_3 = 0. \end{cases} \tag{3.6}$$

Although system (3.6) is rather similar to (2.2), the different signs, due to the different signature of the metric, are responsible for the existence of many more solutions for (3.6). Solving (3.6), we prove the following.

Theorem 3.4. *Consider the three-dimensional unimodular Lorentzian Lie algebra $\mathfrak{g}_3$, as described by* (3.5) *with respect to a suitable pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time-like. Then, the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}_3$ are the following:*

(1) $A = B = C$, $\alpha = 0$, $\lambda = \frac{1}{2}\beta A^2$, for all X: when $A = B = C$, all left-invariant vector fields are Killing.

(2) $A = B - C = 0$, $\alpha = \lambda = 0$, $X_2 = X_3 = 0$: $X = X_1 e_1$ is Killing for the flat metric obtained when $A = B - C = 0$ (corresponding solutions occur for $A - B = C = 0$ and $A - C = B = 0$).

(3) $A = B - C = 0$, $\lambda = 0$, $X = 0$, for all α, β: the metric is flat (corresponding solutions occur for $A - B = C = 0$ and $A - C = B = 0$).

(4) $A = B = C$, $\lambda = \frac{1}{2}\beta A^2$, $X = 0$, for all α, β: the metric is Einstein.

(5) $B=C\neq A$, $\lambda=\frac{1}{2}\beta A(2C-A)$, $X_1=\pm\sqrt{\frac{\beta A(C-A)}{\alpha}}$, $X_2=X_3=0$, *for all* α,β, *with* $\alpha\beta A(C-A)>0$.

(6) $A=B\neq C$, $\lambda=\frac{1}{2}\beta C(2B-C)$, $X_1=X_2=0$, $X_3=\pm\sqrt{\frac{\beta C(C-B)}{\alpha}}$, *for all* α,β, *with* $\alpha\beta C(C-B)>0$.

(7) $A+B=C=0$, $\beta=-\frac{3}{8\alpha}$, $\lambda=\frac{A^2}{2\alpha}$, $X_1=-X_2=\frac{\varepsilon A}{\sqrt{2\alpha}}$, $X_3=\frac{\varepsilon A}{2\alpha}$, $\varepsilon=\pm1$, *for all* $\alpha\neq 0$.

(8) $A=-\frac{\varepsilon(2X_1^2+X_2^2)}{4\beta\sqrt{X_1^2+X_2^2}}$, $B=\frac{\varepsilon(X_1^2+2X_2^2)}{4\beta\sqrt{X_1^2+X_2^2}}$, $C=\frac{\varepsilon(X_1^2-X_2^2)}{4\beta\sqrt{X_1^2+X_2^2}}$, $\alpha=-\frac{3}{8\beta}$, $\lambda=-\frac{X_1^4+X_1^2X_2^2+X_2^4}{4\beta(X_1^2+X_2^2)}$, $X_3=-\frac{\varepsilon X_1X_2}{\sqrt{X_1^2+X_2^2}}$, *for all* $\beta\neq 0$.

Since $\alpha\beta=-\frac{3}{8}$, solutions *(7)* and *(8)* are not compatible with any of equations (RS), (E-W), (PS) and (VN-H). We now focus our attention to solutions *(5)* and *(6)*. Note that these solutions are very similar to one another. However, we listed both of them, since for *(5)* the vector satisfying (3.6) is space-like, while for *(6)* is time-like.

Solutions (5) and (6) are compatible with (E-W), (PS) and (VN-H). More precisely, for any choice of A and $C\neq A$:

- If $A(A-C)>0$, then by *(5)* we have that the same left-invariant Lorentzian metric on $\mathfrak{g}_3$ is solution to both (E-W) and (PS).
- If $A(A-C)<0$, then *(5)* describes a left-invariant Lorentzian metric on $\mathfrak{g}_3$, which is a solution to (VN-H).

Similar observations hold for solution *(6)*, discussing the cases when $\alpha\beta C(C-B)>0$. Hence, taking into account Table II, we have the following result.

Corollary 3.5. *Three-dimensional Lorentzian Lie groups* $\widetilde{SL}(2,\mathbb{R})$ *and* H_3*, with Lie algebra described by* (3.5)*, give solutions to the special Einstein-Weyl equation* (E-W)*, the equation* (PS) *for a metric projective structure with a skew-symmetric Ricci tensor representative and to the vacuum near-horizon geometry equation* (VN-H)*. Three-dimensional Lorentzian Lie group* $SU(2)$*, with Lie algebra described by* (3.5)*, gives solutions to* (VN-H)*.*

3.4. **Lie algebra $\mathfrak{g}_4$.** There exists a pseudo-orthonormal basis $\{e_1,e_2,e_3\}$, with e_3 time-like, such that

$$\mathfrak{g}_4:\quad \begin{aligned} &[e_1,e_2]=-e_2+(2\eta-B)e_3, \qquad \eta=\pm1,\\ &[e_1,e_3]=-Be_2+e_3,\\ &[e_2,e_3]=Ae_1. \end{aligned} \tag{3.7}$$

Table III below describes all Lie groups G admitting a Lie algebra g_4:

Lie group	ηA	B
$\widetilde{SL}(2,\mathbb{R})$	$\neq 0$	$\neq \eta$
$E(1,1)$	0	$\neq \eta$
$E(1,1)$	< 0	η
$\widetilde{E}(2)$	> 0	η
H_3	0	η

Table III: $3D$ *Lorentzian Lie groups with Lie algebra* $\mathfrak{g}_4$

With respect to $\{e_i\}$, the Ricci tensor is described by (see [4])

$$Ric = \begin{pmatrix} -\frac{1}{2}A^2 & 0 & 0 \\ 0 & \frac{1}{2}A^2 + 2\eta(A-B) - AB + 2 & A + 2(\eta - B) \\ 0 & A + 2(\eta - B) & -\frac{1}{2}A^2 + AB + 2 - 2\eta B \end{pmatrix},$$

and the left-invariant metric is Einstein if and only if $A = B - \eta = 0$.

For a left-invariant vector field $X = X_i e_i \in \mathfrak{g}$, we have

$$\mathcal{L}_X g = \begin{pmatrix} 0 & -X_2 + (A-B)X_3 & (B - A - 2\eta)X_2 - X_3 \\ -X_2 + (A-B)X_3 & 2X_1 & 2\eta X_1 \\ (B - A - 2\eta)X_2 - X_3 & 2\eta X_1 & 2X_1 \end{pmatrix}$$

with respect to the basis $\{e_1, e_2, e_3\}$, and $X^\flat \odot X^\flat(e_i, e_j) = \varepsilon_i \varepsilon_j X_i X_j$. Therefore, equation (1.1) now becomes

$$\text{(3.8)} \qquad \begin{cases} 2\alpha X_1^2 + \beta A^2 = 2\lambda, \\ 2X_1 + 2\alpha X_2^2 - \beta(A^2 + 4\eta(A-B) - 2AB + 4) = 2\lambda, \\ 2X_1 + 2\alpha X_3^2 + \beta(A^2 - 2AB - 4 + 4\eta B) = -2\lambda, \\ -X_2 + (A-B)X_3 + 2\alpha X_1 X_2 = 0, \\ (B - A - 2\eta)X_2 - X_3 - 2\alpha X_1 X_3 = 0, \\ 2\eta X_1 - 2\alpha X_2 X_3 - 2\beta(A + 2(\eta - B)) = 0. \end{cases}$$

We then solve (3.8) and prove the following.

Theorem 3.6. *Consider the three-dimensional unimodular Lorentzian Lie algebra* $\mathfrak{g}_4$*, as described by* (3.7) *with respect to a suitable pseudo-orthonormal basis* $\{e_1, e_2, e_3\}$*, with* e_3 *time-like. Then, the nontrivial left-invariant generalized Ricci solitons on* $\mathfrak{g}_4$ *are the following:*

(1) $A = B - \eta$, $\lambda = \frac{1}{2}\beta A^2$, $X_1 = 0$, $X_2 = -\eta X_3$, $X_3 = \pm\sqrt{-\frac{\eta\beta A}{\alpha}}$, *for all* α, β *satisfying* $\eta A \alpha\beta < 0$.

(2) $A = B - \eta$, $\alpha = 0$, $\lambda = \frac{1}{2}\beta A^2$, $X_1 = -\eta\beta A$, $X_2 = -\eta X_3$, *for any value of* β *and* X_3.

(3) $B = \frac{1}{2}A + \eta$, $\beta = -\frac{1}{8\alpha}$, $\lambda = 0$, $X_1 = \frac{\eta A}{4\alpha}$, $X_2 = \pm\sqrt{-\frac{\eta A}{4\alpha^2}}$, $X_3 = -\eta X_2$, *for any* $\alpha \neq 0$ *and* $\eta A < 0$.

(4) $\beta = -\frac{A(A-B+\eta)}{\alpha(A-2B+2\eta)^2}$, $\lambda = -\frac{1}{2}\beta A(A-2B+2\eta)$, $X_1 = \beta(A-2B+2\eta)$, $X_2 = X_3 = 0$, *for any* $\alpha \neq 0$ *and* $A - 2B + 2\eta \neq 0$.

(5) $\beta = -\frac{A-B+\eta}{4\alpha A}$, $\lambda = \frac{(A-B+\eta)(A-2B+2\eta)}{8\alpha}$, $X_1 = \frac{A-B+\eta}{2\eta\alpha}$, $X_2 = -\eta X_3$ *and*

$$X_3 = \pm\frac{1}{2\alpha}\sqrt{\frac{5\eta AB - 3\eta A^2 - 5A - 2\eta + 4B - 2\eta B^2}{A}},$$

for any $\alpha \neq 0$ *and* $A \neq 0$.

In the above Theorem 3.6, solution *(2)* corresponds to the existence of Ricci solitons for Lorentzian Lie algebras of the form $\mathfrak{g}_4$ [1]. Solution *(1)* requires that the sign of $\alpha\beta$ is opposite to the one of ηA. Consequently, taking into account the above Table III, it yields solutions to (E-W) for Lorentzian Lie groups $\widetilde{SL}(2,\mathbb{R})$ and $\widetilde{E}(2)$, and solutions to (VN-H) for $\widetilde{SL}(2,\mathbb{R})$ and $E(1,1)$.

In solution *(4)*, we have $\alpha\beta = -\frac{A(A-B+\eta)}{(A-2B+2\eta)^2}$. For (E-W), as $\alpha\beta = -1$, this equation becomes $4B^2+(5A+8\eta)B+3\eta A = 0$, which admits real solutions for any value of A. On the other hand, for (VN-H) we have $\alpha\beta = \frac{1}{2}$. Hence, we find $3A^2-6(B-\eta)A+4(B-\eta)^2 = 0$, whih has not real solutions. Finally, it is easily seen that solution *(4)* is not compatible with (PS).

By similar arguments, from solution *(5)* we find again $\widetilde{SL}(2,\mathbb{R})$ as solution to (E-W) and (VN-H). Thus, we proved the following.

Corollary 3.7. *Three-dimensional Lorentzian Lie group* $\widetilde{SL}(2,\mathbb{R})$, *with Lie algebra described by* (3.7), *gives solutions to the special Einstein-Weyl equation* (E-W) *and the vacuum near-horizon equation* (VN-H). *Moreover, Lorentzian Lie groups* $\widetilde{E}(2)$ *and* $E(1,1)$, *with Lie algebra described by* (3.7), *give solutions to* (E-W) *and* (VN-H) *respectively.*

4. 3D Lorentzian left-invariant non-unimodular generalized Ricci solitons

Let now $\mathfrak{g}$ denote a three-dimensional non-unimodular Lorentzian Lie algebra. Differently from the Riemannian case, we must now consider three distinct cases, depending on whether the two-dimensional unimodular kernel $\mathfrak{u}$ is either space-like, time-like or degenerate [11]. These three cases were listed in [3] as Lorentzian Lie algebras $\mathfrak{g}_5$, $\mathfrak{g}_6$ and $\mathfrak{g}_7$.

4.1. **Lie algebra $\mathfrak{g}_5$.** There exists a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time-like, such that

(4.1)

$$[e_1, e_2] = 0, \quad [e_1, e_3] = Ae_1 + Be_2, \quad [e_2, e_3] = Ce_1 + De_2, \quad A + D \neq 0, \ AC + BD = 0,$$

for some real constants A, B, C, D.
With respect to the basis $\{e_1, e_2, e_3\}$, the Ricci curvature is described by (see [4])

$$Ric = \begin{pmatrix} A^2 + \frac{1}{2}B^2 - \frac{1}{2}C^2 + AD & 0 & 0 \\ 0 & AD - \frac{1}{2}B^2 + \frac{1}{2}C^2 + D^2 & 0 \\ 0 & 0 & -A^2 - \frac{1}{2}B^2 - \frac{1}{2}C^2 - D^2 - BC \end{pmatrix}.$$

In particular,the left-invariant metric is of constant sectional curvature if and only if $A - D = B + C = 0$.

For an arbitrary left-invariant vector field $X = X_i e_i \in \mathfrak{g}$, we have

$$\mathcal{L}_X g = \begin{pmatrix} 2AX_3 & (B+C)X_3 & -AX_1 - CX_2 \\ (B+C)X_3 & 2DX_3 & -BX_1 - DX_2 \\ -AX_1 - CX_2 & -BX_1 - DX_2 & 0 \end{pmatrix}.$$

With respect to the basis $\{e_1, e_2, e_3\}$, we have again $X^\flat \odot X^\flat(e_i, e_j) = \varepsilon_i \varepsilon_j X_i X_j$. Hence, equation (1.1) now gives

$$\begin{cases} 2AX_3 + 2\alpha X_1^2 - \beta(2A^2 + B^2 - C^2 + 2AD) = 2\lambda, \\ 2DX_3 + 2\alpha X_2^2 - \beta(2AD - B^2 + C^2 + 2D^2) = 2\lambda, \\ 2\alpha X_3^2 + \beta(2A^2 + B^2 + C^2 + 2D^2 + 2BC) = -2\lambda, \\ (B+C)X_3 + 2\alpha X_1 X_2 = 0, \\ -AX_1 - CX_2 - 2\alpha X_1 X_3 = 0, \\ -BX_1 - DX_2 - 2\alpha X_2 X_3 = 0. \end{cases} \tag{4.2}$$

We then solve (4.2) and obtain the following.

Theorem 4.1. *Let $\mathfrak{g}$ denote a three-dimensional non-unimodular Lorentzian Lie algebra $\mathfrak{g}_5$, as described by (4.1) with respect to a suitable pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time-like. Then, the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}_5$ are the following:*

(1) $A - D = B + C = 0$, $\lambda = -(\frac{1}{\alpha} + 2\beta)A^2$, $X_1 = X_2 = 0$, $X_3 = -\frac{A}{\alpha}$, for all $\alpha \neq 0$ and β (constant sectional curvature).

(2) $C = D = 0$, $\beta = -\frac{1}{4\alpha}$, $\lambda = \frac{B^2}{8\alpha}$, $X_1 = \varepsilon X_3$, $X_2 = \frac{\varepsilon B}{2\alpha}$, $X_3 = -\frac{A}{2\alpha}$, $\varepsilon = \pm 1$, for all $\alpha \neq 0$.

(3) $A - D = B + C = 0$, $\lambda = -2\beta A^2$, $X = 0$, for all α, β: the metric is Einstein.

(4) $C = D = 0$, $\lambda = -\frac{1}{2}\beta(2A^2 + B^2)$, $X_1 = X_3 = 0$, $X_2 = \pm\sqrt{-\frac{\beta(A^2+B^2)}{\alpha}}$, for any α and β satisfying $\alpha\beta < 0$.

(5) $B=C=D=0$, $\beta=-\frac{1}{\alpha}$, $\lambda=0$, $X_1=X_2=0$, $X_3=-\frac{A}{\alpha}$, *for any* $\alpha\neq 0$.

(6) $B=C=0$, $\lambda=\frac{A^2(32\alpha^3\beta^3+28\alpha^2\beta^2+9\alpha\beta+1)}{4\alpha(2\alpha\beta+1)^2}$, $X_1=\pm\frac{A\sqrt{8\alpha^2\beta^2+5\alpha\beta+1}}{2\alpha(2\alpha\beta+1)}$, $X_2=0$, $X_3=-\frac{A}{2\alpha}$, *for any* $\alpha\neq 0$ *and* β, *with* $2\alpha\beta+1\neq 0$ *(note that* $a^2\beta^2+5\alpha\beta+1>0$*).*

(7) $B=C=0$, $\beta=-\frac{2A-D}{4\alpha(A-D)}$, $\lambda=-\frac{A(2A^2-AD+D^2)}{4\alpha(A-D)}$, $X_1=0$, $X_2=\pm\frac{1}{\alpha}\sqrt{A^2-\frac{1}{2}AD+\frac{1}{2}D^2}$, $X_3=-\frac{D}{2\alpha}$, *for any* $\alpha\neq 0$ *and* $A\neq D$ *(note that* $2A^2-AD+D^2>0$*).*

It is easy to check that several of the above solutions are compatible with (E-W),(PS) and (VN-H). In particular, solution *(1)* yields solutions to all these equations. Hence, we have the following.

Corollary 4.2. *Three-dimensional non-unimodular Lorentzian Lie groups with Lie algebra* $\mathfrak{g}_5$ *give solutions to the special Einstein-Weyl equation* (E-W)*, to the equation* (PS) *for a metric projective structure with a skew-symmetric Ricci tensor representative and to the vacuum near-horizon geometry equation* (VN-H)*.*

4.2. **Lie algebra $\mathfrak{g}_6$.** There exists a pseudo-orthonormal basis $\{e_1,e_2,e_3\}$, with e_3 time-like, such that

(4.3)
$$[e_1,e_2]=Ae_2+Be_3,\quad [e_1,e_3]=Ce_2+De_3,\quad [e_2,e_3]=0,\quad A+D\neq 0,\ AC-BD=0,$$

for some real constants A,B,C,D.
Following [4], with respect to the basis $\{e_1,e_2,e_3\}$ we have

$$Ric=\begin{pmatrix} \frac{1}{2}B^2-A^2+\frac{1}{2}C^2-D^2-BC & 0 & 0\\ 0 & \frac{1}{2}B^2-A^2-\frac{1}{2}C^2-AD & 0\\ 0 & 0 & AD+\frac{1}{2}B^2-\frac{1}{2}C^2+D^2\end{pmatrix}.$$

In particular, the left-invariant metric is of constant sectional curvature if and only if either $A-D=B-C=0$, $A+B=C+D=0$ or $A-B=C-D=0$.

For an arbitrary left-invariant vector field $X=X_ie_i\in\mathfrak{g}$, we have

$$\mathcal{L}_Xg=\begin{pmatrix} 0 & AX_2+CX_3 & -BX_2-DX_3\\ AX_2+CX_3 & -2AX_1 & (B-C)X_1\\ -BX_2-DX_3 & (B-C)X_1 & 2DX_1\end{pmatrix}.$$

Equation (1.1) now becomes

$$
\begin{cases}
2\alpha X_1^2+\beta(2A^2-B^2-C^2+2D^2+2BC)=2\lambda,\\
-2AX_1+2\alpha X_2^2+\beta(2A^2-B^2+C^2+2AD)=2\lambda,\\
2DX_1+2\alpha X_3^2-\beta(2AD+B^2-C^2+2D^2)=-2\lambda,\\
AX_2+CX_3+2\alpha X_1X_2=0,\\
-BX_2-DX_3-2\alpha X_1X_3=0,\\
(B-C)X_1-2\alpha X_2X_3=0.
\end{cases}
\tag{4.4}
$$

We solve (4.4) and prove the following.

Theorem 4.3. *Let $\mathfrak{g}$ denote a three-dimensional non-unimodular Lorentzian Lie algebra $\mathfrak{g}_6$, as described by* (4.3) *with respect to a suitable pseudo-orthonormal basis $\{e_1,e_2,e_3\}$, with e_3 time-like. Then, the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}_6$ are the following:*

(1) $A-D=B-C=0$, $\lambda=(\frac{1}{\alpha}+2\beta)A^2$, $X_1=-\frac{A}{\alpha}$, $X_2=X_3=0$, for all $\alpha\neq 0$ and β (constant sectional curvature).

(2) $A-D=B-C=0$, $\lambda=2\beta A^2$, $X=0$, for all α and β (Einstein).

(3) $B=\pm A, C=\pm D$, $\lambda=\frac{1}{2}\beta(A+D)^2$, $X=0$, for all α,β (Einstein).

(4) $C=D=0$, $\lambda=\frac{1}{2}\beta(2A^2-B^2)$, $X_1=X_2=0$, $X_3=\pm\sqrt{\frac{\beta(B^2-A^2)}{\alpha}}$, for any α and β satisfying $\alpha\beta(B^2-A^2)>0$.

(5) $B=C=0$, $\beta=-\frac{A^2+D^2}{\alpha(A+D)^2}$, $\lambda=0$, $X_1=-\frac{A^2+D^2}{\alpha(A+D)}$, $X_2=X_3=0$, for any $\alpha\neq 0$.

(6) $B=C=0$, $\beta=-\frac{2A-D}{4\alpha(A-D)}$, $\lambda=-\frac{A(2A^2-AD+D^2)}{4\alpha(A-D)}$, $X_1=-\frac{D}{2\alpha}$, $X_2=0$, $X_3=\pm\frac{1}{2\alpha}\sqrt{2A^2-AD+D^2}$, $X_3=0$, for any $\alpha\neq 0$ and $A\neq D$ (note that $2A^2-AD+D^2>0$).

(7) $A=B=0$, $\lambda=\frac{1}{2}\beta(2D^2-C^2)$, $X_1=0$, $X_2=\pm\sqrt{\frac{\beta(D^2-C^2)}{\alpha}}$, $X_3=0$, for all α,β with $\alpha\beta(D^2-C^2)>0$.

By the same argument used in the previous case, it is easy to check that several of the above solutions are compatible with (E-W),(PS) and (VN-H). Thus, we have the following.

Corollary 4.4. *Three-dimensional non-unimodular Lorentzian Lie groups with Lie algebra $\mathfrak{g}_6$ give solutions to the special Einstein-Weyl equation* (E-W)*, to the equation* (PS) *for a metric projective structure with a skew-symmetric Ricci tensor representative and to the vacuum near-horizon geometry equation* (VN-H)*.*

4.3. **Lie algebra $\mathfrak{g}_7$.** There exists a pseudo-orthonormal basis $\{e_1,e_2,e_3\}$, with e_3 time-like, such that

$$
[e_1,e_2]=-[e_1,e_3]=-Ae_1-Be_2-Be_3,\quad [e_2,e_3]=Ce_1+De_2+De_3,\quad A+D\neq 0,\ AC=0,
\tag{4.5}
$$

for some real constants A, B, C, D. The Ricci curvature is then described by (see [4])

$$Ric = \begin{pmatrix} -\frac{1}{2}C^2 & 0 & 0 \\ 0 & AD - A^2 + \frac{1}{2}C^2 - BC & A^2 - AD + BC \\ 0 & A^2 - AD + BC & AD - A^2 - \frac{1}{2}C^2 - BC \end{pmatrix}$$

and the left-invariant metric is of constant sectional curvature (flat) if and only if either $A = C = 0$ or $A - D = C = 0$. For a vector $X = X_i e_i \in \mathfrak{g}$, we have

$$\mathcal{L}_X g = \begin{pmatrix} -2A(X_2 - X_3) & AX_1 - BX_2 + (B+C)X_3 & -AX_1 + (B-C)X_2 - BX_3 \\ AX_1 - BX_2 + (B+C)X_3 & 2BX_1 + 2DX_3 & -2BX_1 - DX_2 - DX_3 \\ -AX_1 + (B-C)X_2 - BX_3 & -2BX_1 - DX_2 - DX_3 & 2BX_1 + 2DX_2 \end{pmatrix}.$$

Thus, equation (1.1) becomes

$$\begin{cases} -2A(X_2 - X_3) + 2\alpha X_1^2 + \beta C^2 = 2\lambda, \\ 2BX_1 + 2DX_3 + 2\alpha X_2^2 + \beta(2A^2 - 2AD - C^2 + 2BC) = 2\lambda, \\ 2BX_1 + 2DX_2 + 2\alpha X_3^2 + \beta(2A^2 - 2AD + C^2 + 2BC) = -2\lambda, \\ AX_1 - BX_2 + (B+C)X_3 + 2\alpha X_1 X_2 = 0, \\ -AX_1 + (B-C)X_2 - BX_3 - 2\alpha X_1 X_3 = 0, \\ -2BX_1 - DX_2 - DX_3 - 2\alpha X_2 X_3 - 2\beta(A^2 - AD + BC) = 0. \end{cases} \tag{4.6}$$

Solving (4.6), we prove the following.

Theorem 4.5. *Let $\mathfrak{g}$ denote a three-dimensional non-unimodular Lorentzian Lie algebra $\mathfrak{g}_7$, as described by (4.5) with respect to a suitable pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time-like. Then, the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}_7$ are the following:*

(1) $A = C = 0$, $\alpha = 0$, $\lambda = 0$, $X_2 = X_3 = -\frac{B}{D}X_1$, for all β and X_1 (flat).

(2) $A = \frac{1}{2}D$, $C = 0$, $\alpha = 0$, $X_1 = -\frac{4B\lambda}{D^2}$, $X_2 = \frac{16\lambda B^2 - 4\lambda D^2 + \beta D^4}{4D^3}$, $X_3 = \frac{16\lambda B^2 + 4\lambda D^2 + \beta D^4}{4D^3}$, for all β and λ.

(3) $C = 0$, $\lambda = 0$, $X_1 = 0$, $X_2 = X_3$ solution of

$$\alpha x^2 + Dx + \beta A(A - D) = 0,$$

for all A, B, D and α, β such that $D^2 - 4\alpha\beta A(A - D) > 0$.

(4) either $A = C = 0$ or $A - D = C = 0$, $\lambda = 0$, $X = 0$: flat metric.

(5) $A = \frac{1}{2}D$, $B = C = 0$, $\alpha = 0$ $\lambda = \beta A^2$, $X_1 = X_2 = 0$, $X_3 = \beta A$, for any $\beta \neq 0$.

(6) $A = 0$, $\beta = -\frac{1}{4\alpha}$ $\lambda = \frac{C^2}{8\alpha}$, $X_1 = -\frac{C}{2\alpha}$, $X_2 = X_3$ solution of

$$4\alpha^2 x^2 + 4\alpha Dx - 3BC = 0,$$

for any $\alpha \neq 0$, whenever $D^2 + 3BC > 0$.

(7) $A = B = 0$, $\lambda = -\frac{1}{2}\beta C^2$, $X_1 = \pm\sqrt{\frac{-\beta C^2}{\alpha}}$, $X_2 = X_3 = 0$, *for any* α, β *with* $\alpha\beta C^2 < 0$.

(8) $A = 0$, $\beta = -\frac{1}{\alpha}$, $\lambda = -\frac{1}{2}\beta C^2$, $X_1 = -\beta C$, $X_2 = X_3 = 0$, *for any* $\alpha \neq 0$.

The above solutions with $\alpha = 0$ correspond to the existence of left-invariant Ricci solitons on $\mathfrak{g}_7$ [1]. Moreover, several of the above solutions are compatible with (E-W), (PS) and (VN-H). In particular, whatever the value of α and β, if D is sufficiently big then $D^2 - 4\alpha\beta A(A-D) > 0$. Henceforth, *(3)* yields solutions to all (E-W), (PS) and (VN-H). Thus, we have the following.

Corollary 4.6. *Three-dimensional non-unimodular Lorentzian Lie groups with Lie algebra* $\mathfrak{g}_7$ *give solutions to the special Einstein-Weyl equation* (E-W)*, to the equation* (PS) *for a metric projective structure with a skew-symmetric Ricci tensor representative and to the vacuum near-horizon geometry equation* (VN-H)*.*

5. Generalized Ricci solitons on special 3D Lie groups

Let us consider a Lie algebra $\mathfrak{g}$ (of dimension ≥ 2) with the property that there exists a linear map $l : \mathfrak{g} \to \mathbb{R}$, such that

$$[x, y] = l(x)y - l(y)x, \qquad x, y \in \mathfrak{g}, \tag{5.1}$$

that is, the bracket product $[x, y]$ is always a linear combination of x and y. Left-invariant Riemannian metrics on a Lie algebra described by (5.1) were considered in [13], showing that they have constant sectional curvature $K = -||l||^2$ (thus, negative, unless $\mathfrak{g}$ is abelian). Left-invariant Lorentzian metrics on Lie algebra (5.1) were investigated in [14]. Again, they are of constant sectional curvature K, but K can attain any real value.

For a three-dimensional Lie algebra $\mathfrak{g}$, with respect to any basis $\{e_1, e_2, e_3\}$, from (5.1) we get

$$[e_1, e_2] = Be_1 - Ae_2, \quad [e_1, e_3] = Ce_1 - Ae_3, \quad [e_2, e_3] = Ce_2 - Be_3, \tag{5.2}$$

for three real constants $A = l(e_1), B = l(e_2), C = l(e_3)$. When we consider a left-invariant Riemannian metric g on $\mathfrak{g}$, we take $\{e_1, e_2, e_3\}$ as an orthonormal basis. Proceeding as in Section 2, we can then describe the Ricci tensor and $\mathcal{L}_X g$ with respect to $\{e_1, e_2, e_3\}$, for any left-invariant vector $X = X_i e_i$. In this way, (1.1) yields

$$\begin{cases} BX_2 + CX_3 + \alpha X_1^2 + 2\beta(A^2 + B^2 + C^2) = \lambda, \\ AX_1 + CX_3 + \alpha X_2^2 + 2\beta(A^2 + B^2 + C^2) = \lambda, \\ AX_1 + BX_2 + \alpha X_3^2 + 2\beta(A^2 + B^2 + C^2) = \lambda, \\ -BX_1 - AX_2 + 2\alpha X_1 X_2 = 0, \\ -CX_1 - AX_3 + 2\alpha X_1 X_3 = 0, \\ -CX_2 - BX_3 + 2\alpha X_2 X_3 = 0. \end{cases} \tag{5.3}$$

By a similar argument, for a left-invariant Lorentzian metric g on $\mathfrak{g}$, we take a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time-like, and $\mathfrak{g}$ is again described by (5.2). We then proceed as in Section 3 and find that for a left-invariant vector $X = X_i e_i$, equation (1.1) now becomes

$$\begin{cases} BX_2 + CX_3 + \alpha X_1^2 + 2\beta(A^2 + B^2 - C^2) = \lambda, \\ AX_1 + CX_3 + \alpha X_2^2 + 2\beta(A^2 + B^2 - C^2) = \lambda, \\ -AX_1 - BX_2 + \alpha X_3^2 - 2\beta(A^2 + B^2 - C^2) = -\lambda, \\ -BX_1 - AX_2 + 2\alpha X_1 X_2 = 0, \\ -CX_1 + AX_3 - 2\alpha X_1 X_3 = 0, \\ -CX_2 + BX_3 - 2\alpha X_2 X_3 = 0. \end{cases} \tag{5.4}$$

Only some changes of sign occur between systems (5.3) and (5.4), due to the different signatures of the metrics. However, these slight changes are responsible, once again, of differences concerning their solutions. Solving (5.3) and (5.4), we prove the following.

Theorem 5.1. *Let $\mathfrak{g}$ denote a three-dimensional Lie algebra described by* (5.1)*.*

(I) If g is a left-invariant Riemannian metric on $\mathfrak{g}$, then (5.2) *holds with respect to an orthonormal basis $\{e_1, e_2, e_3\}$, and the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}$ are the following:*

(1) $\lambda = 2\beta(A^2 + B^2 + C^2)$, $X = 0$: the metric is Einstein.

(2) $A = B = C = 0$, $\alpha = 0$, $\lambda = 0$, X is arbitrary: abelian Lie algebra.

(3) $\lambda = (\frac{1}{\alpha} + 2\beta)(A^2 + B^2 + C^2)$, $X_1 = \frac{A}{\alpha}$, $X_2 = \frac{B}{\alpha}$, $X_3 = \frac{C}{\alpha}$.

(II) If g is a left-invariant Lorentzian metric on $\mathfrak{g}$, then (5.2) *holds with respect to a pseudo-orthonormal basis $\{e_1, e_2, e_3\}$, with e_3 time-like, and the nontrivial left-invariant generalized Ricci solitons on $\mathfrak{g}$ are the following:*

(1') $\lambda = 2\beta(A^2 + B^2 - C^2)$, $X = 0$: the metric is Einstein.

(2') $A = B = C = 0$, $\alpha = 0$, $\lambda = 0$, X is arbitrary: abelian Lie algebra.

(3') either $A = B = 0$ or $C = 0$, and $\lambda = (\frac{1}{\alpha} + 2\beta)(A^2 + B^2 - C^2)$, $X_1 = \frac{A}{\alpha}$, $X_2 = \frac{B}{\alpha}$, $X_3 = -\frac{C}{\alpha}$.

It may be observed that solution *(3)* of the Riemannian case corresponds to solution *(3')* of the Lorentzian one, which holds either for a space-like vector $X = \frac{A}{\alpha}e_1 + \frac{B}{\alpha}e_2$, or for a time-like vector $X_3 = -\frac{C}{\alpha}e_3$.

It is easily seen that solutions *(3)* and *(3')* are compatible with (E-W) and (VN-H) and also with (PS), as $\frac{1}{\alpha} + 2\beta = 0$ when $\alpha = 1$ and $\beta = -\frac{1}{2}$. Therefore, the following result holds.

Corollary 5.2. *Three-dimensional Lie groups, with special Lie algebra* (5.1)*, give solutions to the special Einstein-Weyl equation* (E-W)*, to the equation* (PS) *for a metric*

projective structure with a skew-symmetric Ricci tensor representative and to the vacuum near-horizon geometry equation (VN-H).

6. Final remarks

We end with some observations concerning the geometry of some of the solutions we found to the generalized Ricci soliton equation (1.1) for three-dimensional Lie groups.

The most interesting case occurring in Theorem 2.1, namely, solution *(4)*, requires the structure coefficients A, B, C to satisfy $A = B \neq C$ (or any of their permutation). This condition occurs in the classification of naturally reductive Riemannian three-manifolds [19]. More precisely, for $C = 0$ one gets a flat metric on $\widetilde{E}(2)$, while for $C \neq 0$, one finds the proper (that is, not locally symmetric) examples of naturally reductive spaces as left-invariant metrics on $SU(2)$, $\widetilde{SL}(2, \mathbb{R})$ and H_3. Correspondingly, naturally reductive Lorentzian three-manifolds are found in the cases of Lie algebra $\mathfrak{g}_3$ satisfying $A = B \neq C$ (with $C \neq 0$ in the proper case), and of Lie algebra $\mathfrak{g}_4$ satisfying $A = B - \eta$ (with $A \neq 0$ in the proper case) (see [5],[6]). Again, these conditions appear in the solutions to the generalized Ricci soliton equation for these Lorentzian Lie algebras, in Theorems 3.4 and 3.6, respectively.

If a Riemannian manifold (M^n, g) admits a parallel line field $\mathcal{D}$, then (M^n, g) splits locally into the direct product of a line and an $(n-1)$-dimensional manifold. The same property is true for a pseudo-Riemannian manifold admitting a parallel line field $\mathcal{D}$, either space-like or time-like. However, in pseudo-Riemannian settings, a different phenomenon can occur: it may exist a parallel degenerate line field $\mathcal{D}$, locally spanned by a light-like vector field U satisfying $\nabla U = \omega \otimes U$. Pseudo-Riemannian manifolds admitting a parallel vector field are a special case of the so called *Walker manifolds*, and are responsible for several behaviours which do not have any Riemannian counterpart. We may refer to [9] for the investigation of three-dimensional Walker manifolds.

With regard to Lie algebra $\mathfrak{g}_1$, when $B = 0$ (that is, for Lorentzian Lie group $E(1,1)$), vector $X = X_2(e_2 + e_3)$ is light-like and ∇X is parallel to X, for any real constant $X_2 \neq 0$. Therefore, this is a Walker manifold, and a vector of the form $X = X_2(e_2 + e_3)$ occurs in solution *(3)* of Theorem 3.1 for the generalized Ricci soliton equation. For non-unimodular Lie algebra $\mathfrak{g}_7$, whenever $C = 0$, the Lorentzian metric is again Walker (with both locally symmetric and not locally symmetric examples occurring), as ∇X is parallel to X when $X = X_2(e_2 + e_3)$. And a vector of this form appears in solution *(3)* of Theorem 4.5 for the generalized Ricci soliton equation.

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